Math 580: Combinatorics Notes

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Contents

1 Matching in Bipartite Graphs

1.1 Definitions

Definition 0.1 (Matching). A **matching** in a graph G is a set of edges $M \subseteq E(G)$ such that no two edges share a vertex (they are independent).

- A vertex is **covered** (or saturated) by M if it is an endpoint of an edge in M.
- A **perfect matching** (or 1-factor) is a matching that covers all vertices of G.
- The **matching number** $\alpha'(G)$ is the maximum size of a matching in G.

Remark 0.1 (Context). Matchings model one-to-one pairings. In bipartite graphs, this models assigning jobs to applicants or pairing compatible items from two distinct groups. Our goal is often to maximize the size of the matching or check for the existence of a perfect one.

1.2 Hall's Condition

Consider a bipartite graph with parts X, Y. We ask: when does there exist a matching that covers all vertices in X?

Definition 0.2 (Hall's Condition). For a set $S \subseteq X$, let $N(S)$ denote the set of neighbors of S in Y. **Hall's Condition** is satisfied if for every subset $S \subseteq X$,

$$|N(S)| \geq |S|.$$

Remark 0.2 (Intuition). This is an obvious necessary condition. If a subset of k people (S) collectively qualify for fewer than k jobs ($N(S)$), it is impossible to assign a distinct job to each person. This "bottleneck" prevents a matching. Hall's Theorem states that this obvious obstruction is the *only* obstruction.

Theorem 0.1 (Hall's Theorem 1935). Let G be an X, Y-bigraph. G has a matching covering X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

Proof. (**Necessity**) Discussed in the intuition above.

(**Sufficiency**) We proceed by induction on $|X|$. Base case $|X| = 1$ is trivial. For $|X| > 1$, consider two cases regarding the "tightness" of the condition.

Case 1: Strong Inequality. Suppose $|N(S)| \geq |S| + 1$ for all nonempty proper subsets $S \subsetneq X$. Pick any edge xy with $x \in X$. Consider $G' = G - \{x, y\}$. For any $S' \subseteq X \setminus \{x\}$, its neighborhood in G' is $N_{G'}(S') = N_G(S') \setminus \{y\}$. Since $|N_G(S')| \geq |S'| + 1$, we have $|N_{G'}(S')| \geq |S'|$. By IH, G' has a matching covering $X \setminus \{x\}$. Adding xy completes the matching.

Case 2: Equality Exists. Suppose there exists a proper subset $S \subsetneq X$ such that $|N(S)| = |S|$. Let $G_1 = G[S \cup N(S)]$ and $G_2 = G - (S \cup N(S))$.

1. *Matching in G_1:* S satisfies Hall's condition in G_1 (trivial), so by IH (since $|S| < |X|$), G_1 has a matching covering S.
2. *Matching in G_2:* We check Hall's condition for G_2. Let $T \subseteq X \setminus S$. The neighbors of T in G_2 are $N_G(T) \setminus N(S)$. We know $|N_G(T \cup S)| \geq |T \cup S| = |T| + |S|$. Also, $N_G(T \cup S) = N_{G_2}(T) \cup N(S)$. Thus, $|N_{G_2}(T)| + |N(S)| \geq |T| + |S|$. Since $|N(S)| = |S|$, we get $|N_{G_2}(T)| \geq |T|$. By IH, G_2 has a matching covering $X \setminus S$.

The union of these two matchings covers X. □

1.3 Min-Max Relations

Hall's Theorem leads to a formula for the maximum size of a matching even when we cannot cover all of X.

Definition 0.3 (Defect). Let $\alpha'(G)$ denote the maximum size of a matching. For an X, Y-bigraph G, the **defect** of a set $S \subseteq X$ is

$$df(S) = |S| - |N(S)|.$$

Corollary 0.2 (Ore's Defect Formula 1955)**.** In an X, Y-bigraph G, the maximum size of a matching is given by

$$\alpha'(G) = \min_{S \subseteq X} \{|X| - df(S)\}.$$

Proof. Let $d = \max_{S \subseteq X} df(S)$. First, $\alpha'(G) \leq |X| - df(S)$ for any S, because any matching must miss at least $|S| - |N(S)|$ vertices of S (by Pigeonhole Principle). Thus $\alpha'(G) \leq |X| - d$.

We construct a matching of size $|X| - d$. Form a new graph G' by adding d vertices to Y, each connected to all vertices in X. We verify Hall's Condition for G'. For any $S \subseteq X$:

$$|N_{G'}(S)| = |N_G(S)| + d = |S| - df(S) + d.$$

Since $d \geq df(S)$, we have $|N_{G'}(S)| \geq |S|$. Thus G' has a matching covering X (size $|X|$). Removing the d added vertices destroys at most d edges of this matching. The remaining edges form a matching in G of size at least $|X| - d$. □

Definition 0.4. Below are some more basic notions of graph parameters.

- **Vertex Cover** ($\beta(G)$): A set of vertices such that every edge is incident to at least one vertex in the set.
- **Edge Cover** ($\beta'(G)$): A set of edges that covers every vertex (no isolated vertices).
- **Independence Number** ($\alpha(G)$): Max size of a set of pairwise non-adjacent vertices.

Theorem 0.3 (König-Egerváry Theorem 1931)**.** If G is bipartite, then the maximum matching size equals the minimum vertex cover size:

$$\alpha'(G) = \beta(G).$$

Proof. (Easy direction $\beta \geq \alpha'$): Edges in a matching are disjoint, so a vertex cover must pick at least one distinct vertex for each matching edge.

(Hard direction $\beta \leq \alpha'$): We seek a vertex cover of size $\alpha'(G)$. By Ore's Defect Formula, there is a set $T \subseteq X$ such that $\alpha'(G) = |X| - df(T) = |X| - (|T| - |N(T)|) = |X \setminus T| + |N(T)|$.

Consider the set of vertices $Q = (X \setminus T) \cup N(T)$. This set Q covers all edges:

- Edges incident to $X \setminus T$ are covered by $X \setminus T$.
- Edges incident to T must have their other endpoint in $N(T)$, so they are covered by $N(T)$.

The size of this cover is $|Q| = |X \setminus T| + |N(T)| = \alpha'(G)$. Thus $\beta(G) \leq \alpha'(G)$. □

Theorem 0.4 (König's Other Theorem 1916)**.** If G is a bipartite graph with no isolated vertices, then $\alpha(G) = \beta'(G)$. (Maximum Independent Set = Minimum Edge Cover).

Proof. Let Q be a minimum vertex cover. Let $R = X \cap Q$ and $S = Y \cap Q$. By König-Egerváry, $|Q| = \alpha'(G)$, so let M be a maximum matching of size $|Q|$. Since Q covers every edge, M cannot have edges disjoint from Q. In fact, M matches R into $Y \setminus S$ and S into $X \setminus R$.

We construct an edge cover. Take the edges of M. Each edge covers two vertices. The vertices not covered by M are exactly $(X \setminus R) \setminus N_M(S)$ and $(Y \setminus S) \setminus N_M(R)$. Since G has no isolated vertices, we can add one edge for each unmatched vertex to cover it.

The size of this edge cover is:

$$|M| + (n - 2|M|) = n - |M| = n - \alpha'(G).$$

However, for any graph, $\alpha(G) = n - \beta(G)$ (complement of min vertex cover is max independent set). Since $\beta(G) = \alpha'(G)$ (König-Egerváry), we have:

$$\alpha(G) = n - \alpha'(G) = \text{Size of Edge Cover constructed.}$$

Thus $\alpha(G) = \beta'(G)$. □

Remark 0.3 (Gallai's Theorem 1959)**.** The relationship used in the proof above actually holds for **any** graph G without isolated vertices:

$$\alpha'(G) + \beta'(G) = n.$$

(Note: $\alpha(G) + \beta(G) = n$ is trivial for all graphs).

2 Matching in General Graphs

For general graphs, Hall's Condition is not sufficient (e.g., C_3). We need to account for parity.

2.1 Berge-Tutte Formula

We will establish a formula for the maximum matching size in any graph. First, we need some definitions and lemmas.

Definition 0.5 (Deficiency and Tutte's Condition)**.** Let $o(H)$ be the number of odd components of a graph H. The **deficiency** of a set $S \subseteq V(G)$ is:

$$def(S) = o(G - S) - |S|.$$

Tutte's Condition is the statement that $o(G-S) \leq |S|$ for all S. A set with positive deficiency is a **Tutte set**.

Lemma 0.5 (Parity Lemma)**.** If S is a set of vertices in an n-vertex graph G, then

$$o(G-S) - |S| \equiv n \pmod{2}.$$

Proof. $|V(G)| = |S| + _{C \in G-S} |V(C)|$. Modulo 2, $|V(C)| \equiv 1$ if C is odd and 0 if C is even. Thus $n \equiv |S| + o(G-S) \pmod{2} \implies o(G-S) - |S| \equiv n \pmod{2}$. □

Lemma 0.6 (Maximal Set Lemma)**.** Let T be a maximal subset of $V(G)$ among those having largest deficiency.

1. $G - T$ has no even components.
2. If u is a vertex of an odd component C of $G - T$, then $C - u$ satisfies Tutte's Condition (i.e., has a perfect matching).

Proof. (sorry but this particular proof is mostly a copy-paste from textbook) Subscripts on def denote the relevant graph. For $S \subseteq V(C-u)$, the odd components of $C - u - S$ are in $C - u$. Comparing $o(G - T - u - S)$ to $o(G-T)$, we lose one odd component (C) and gain odd components in $C - u - S$. Thus,

$$\begin{aligned} \text{def}_G(T \cup \{u\} \cup S) &= o(G - T - u - S) - (|T| + 1 + |S|) \\ &= o(G-T) - 1 + o(C - u - S) - |T| - 1 - |S| \\ &= \text{def}_G(T) - 2 + \text{def}_{C-u}(S). \end{aligned}$$

Since T is a maximal set of maximum deficiency and $T \cup \{u\} \cup S$ contains T properly (unless $S = \emptyset$ and we just added u), we must have $\text{def}_G(T \cup \{u\} \cup S) < \text{def}(T)$ (if equality held, T wouldn't be maximal). Thus $\text{def}_{C-u}(S) < 2$. Since $C - u$ has even order (as C is odd), $\text{def}_{C-u}(S)$ is even by the Parity Lemma. We conclude that $\text{def}_{C-u}(S) \leq 0$ for all $S \subseteq V(C-u)$, which is Tutte's Condition.

If $G - T$ has an even component C, then let $T' = T \cup \{v\}$, where v is a leaf of a spanning tree of C. We have $|T'| = |T| + 1$. Deleting v from the even component C leaves an odd component (or multiple odd components totaling odd size, but specifically removing a leaf from a connected graph leaves a connected graph), so $o(G-T') = o(G-T)+1$. Hence $\mathrm{def}_G(T') = (o(G-T)+1) - (|T|+1) = \mathrm{def}_G(T)$. This contradicts the choice of T (since T was chosen to be a *maximal* set among those with maximum deficiency). ☐

Theorem 0.7 (Berge-Tutte Formula 1958)**.** In an n-vertex graph G, the maximum number of vertices covered by a matching is $n-d$, where $d = \max_{S\subseteq V(G)} def(S)$. That is:

$$\alpha'(G) = \min_{S\subseteq V(G)} \frac{1}{2}(n - def(S)).$$

Proof. We know $\alpha'(G) \le \frac{1}{2}(n - def(S))$ for all S, because if we delete S, we are left with $o(G-S)$ odd components. Each odd component must have at least one vertex unmatched or matched to S. Since S can cover at most $|S|$ such components, at least $o(G-S)-|S|$ vertices must remain unmatched.

We prove equality by induction on n. Trivial for $n=0$. Let T be a maximal set with maximum deficiency d. By the Maximal Set Lemma, $G-T$ consists only of odd components $C_1, \ldots, C_k$, and for any $u \in C_i$, $C_i - u$ has a perfect matching. Note that $k = o(G-T)$. Also $def(T) = k - |T| = d$, so $k = |T| + d$.

Strategy: We want to match vertices of T to distinct odd components. Since there are $|T|+d$ components, this will leave exactly d components unmatched. In the matched components, one vertex is covered by T, the rest by a perfect matching in $C_i - u$. In the d unmatched components, we pick a perfect matching for $C_i - v$, leaving 1 vertex (v) exposed. Total exposed: d.

Construct an auxiliary bipartite graph H with parts T and $\mathcal{C} = \{C_1, \ldots, C_k\}$. Edge tC_i exists if t is connected to any vertex in C_i in G. We verify Hall's Condition for H to match T into $\mathcal{C}$. For any $S \subseteq T$, let $N_H(S)$ be the components connected to S. The components NOT in $N_H(S)$ are disjoint from S, meaning they are odd components of $G-(T\setminus S)$. Thus:

$$|\mathcal{C} \setminus N_H(S)| \le o(G - (T\setminus S)).$$

Using definition of deficiency: $o(G-(T\setminus S)) = def(T\setminus S) + |T\setminus S|$. Since T has max deficiency d, $def(T\setminus S) \le d$.

$$|\mathcal{C}| - |N_H(S)| \le d + |T| - |S|.$$

Substitute $|\mathcal{C}| = |T| + d$:

$$(|T|+d) - |N_H(S)| \le d + |T| - |S| \implies |N_H(S)| \ge |S|.$$

By Hall's Theorem, we can match all of T into distinct components. The construction follows. ☐

2.2 Tutte's 1-Factor Theorem

Tutte's famous characterization is now a direct corollary of the Berge-Tutte Formula.

Theorem 0.8 (Tutte's 1-Factor Theorem 1947)**.** A graph G has a 1-factor if and only if $o(G-S) \le |S|$ for all $S \subseteq V(G)$.

Proof. G has a 1-factor iff $\alpha'(G) = n/2$. Using Berge-Tutte: $\alpha'(G) = \frac{1}{2}(n - \max def(S))$. So we need $\max def(S) = 0$. This implies $o(G-S) - |S| \le 0$, or $o(G-S) \le |S|$ for all S. ☐

2.3 Regular Graphs

Corollary 0.9 (Petersen 1891)**.** Every 3-regular graph with no cut-edge has a 1-factor.

Proof. We verify Tutte's condition. Let $S \subseteq V(G)$. Let m be the number of edges between S and $G - S$. Let C_i be the odd components of $G - S$. Since G is 3-regular and $|C_i|$ is odd, the sum of degrees in C_i is odd. The internal edges contribute even degree sum, so there must be an odd number of edges leaving C_i. Since there are no cut-edges, at least 3 edges leave each C_i. Thus $m \geq 3 \cdot o(G - S)$. Also, edges must land in S, so $m \leq 3|S|$. Therefore $3o(G - S) \leq 3|S| \implies o(G - S) \leq |S|$. □

Theorem 0.10 (Petersen's 2-Factor Theorem)**.** Every regular graph of positive even degree ($2k$-regular) has a 2-factor.

Proof. Since the graph is even-regular, it is Eulerian. Let C be an Eulerian tour. Construct a bipartite graph H by splitting each vertex v into v_{in} and v_{out}. If the tour goes $u \to v$, add edge $u_{out}v_{in}$. H is a k-regular bipartite graph. By the Marriage Theorem, H has a perfect matching. This matching selects one incoming edge and one outgoing edge for each vertex in G, forming a collection of cycles (a 2-factor). □

3 Connectivity Parameters

3.1 Vertex Connectivity

Definition 0.6 (Vertex cut)**.** Let G be a graph. A set $S \subseteq V(G)$ is called a *cutset* (or *separating set* or *vertex cut*) if the graph $G - S$ has more than one component.

Definition 0.7 (k-connected graph)**.** A graph G is *k-connected* if $|V(G)| > k$ and every vertex cut has size at least k.

Definition 0.8 (Connectivity)**.** The *connectivity* of G is denoted by $\kappa(G)$, and is the maximum integer k such that G is k-connected.

Remark 0.4 (Explanation)**.** Throughout, G is a simple graph. We record some basic facts:

- For the complete graph K_n, we have $\kappa(K_n) = n - 1$.
- For every graph G, $\kappa(G) \leq \delta(G)$, since deleting the neighborhood of a minimum-degree vertex will make the graph disconnected.
- The graph K_1 is an exception: it is connected but $\kappa(K_1) = 0$.
- For a complete bipartite graph $K_{r,s}$, $\kappa(K_{r,s}) = \min\{r, s\}$.
- For a cycle C_n, we have $\kappa(C_n) = 2$.

Definition 0.9 (Harary graph $H_{k,n}$)**.** Let $2 \leq k < n$. The *k-connected Harary graph $H_{k,n}$* is constructed as follows:

- Place the n vertices on a circle and make each vertex adjacent to the nearest $\lfloor k/2 \rfloor$ vertices in both directions.
- If k is odd and n is even, add a "diagonal" edge from vertex i to vertex $i + \frac{n}{2}$ (indices taken modulo n) for each $1 \leq i \leq \frac{n}{2}$ to compensate for the loss.

Remark 0.5 (Note)**.** Here we make some observations on the properties of Harary Graph.

(i) If kn is even, then $H_{k,n}$ is k-regular.

(ii) If kn is odd, then all but one vertex of $H_{k,n}$ have degree $k+1$, and the remaining vertex has degree k.

(iii) The graph $H_{k,n}$ is k-connected for all valid k, n, and $H_{k,n}$ has the **minimum number of edges** among all k-connected graphs on n vertices.

Definition 0.10 (Vertex k-split)**.** Let G be a graph and $x \in V(G)$. A *vertex k-split* of G at x is a graph H obtained from G by replacing the vertex x with two new vertices x_1, x_2 such that:

$$N_H(x_1) \cup N_H(x_2) = N_G(x) \quad \text{and} \quad d_H(x_i) \geq k \text{ for } i = 1, 2.$$

(Each neighbor of x in G is adjacent to at least one of x_1 or x_2 in H.)

Lemma 0.11. If G is k-connected and H is obtained from G by a vertex k-split at some vertex x, then H is also k-connected.

Proof. Let S be a vertex cut in H. We want to show $|S| \geq k$. Let $X = \{x_1, x_2\}$ be the pair of new vertices in H.

Case (i). $\mathbf{S \cap X =}$. Then S is also a vertex cut of G, so $|S| \geq k$ since G is k-connected. Hence H is k-connected in this case.

Case (ii). $\mathbf{X \subseteq S}$. Let $T = (S \setminus X) \cup \{x\} \subseteq V(G)$. By construction, we have $H - S = G - T$, so T is a vertex cut in G. Therefore $|T| \geq k$ and $|S| > |T| \geq k$.

Case (iii). $\mathbf{|S \cap X| = 1}$. Without loss of generality, assume $x_1 \in S$ and $x_2 \notin S$. Set $T = (S \setminus \{x_1\}) \cup \{x\} \subseteq V(G)$.

If T separates G, then $|S| = |T| \geq k$, as required. Otherwise, assume T does *not* separate G, so $G - T$ is connected. We can obtain $H - S$ from $G - T$ by adding back the vertex x_2 and all edges joining x_2 to its neighbors in H. Thus $H - S$ is connected unless S contains all neighbors of x_2, i.e., $N_H(x_2) \subseteq S$. But by construction $d_H(x_2) \geq k$, so if $N_H(x_2) \subseteq S$, then $|S| \geq |N_H(x_2)| \geq k$. Therefore, in every case, any vertex cut S in H satisfies $|S| \geq k$. Hence H is k-connected. □

Theorem 0.12 (General product bound)**.** Let G and H be connected graphs. Then their Cartesian product satisfies

$$\kappa(G \square H) \;\geq\; \kappa(G) + \kappa(H).$$

Next we discuss edge connectivity. Notice how the notion of disconnecting set is formed differently then vertex connectivity. Also notice that the redundancy of multiedges can be valuable in this setting.

3.2 Edge Connectivity

Definition 0.11 (Disconnecting set of edges)**.** Let G be a multigraph. A set F of edges in G is called a *disconnecting set* if $G - F$ is disconnected.

Definition 0.12 (k-edge-connected graph)**.** A graph G is *k-edge-connected* if every disconnecting set $F \subseteq E(G)$ has size at least k; i.e. if $|F| \geq k$ for all disconnecting sets F.

Definition 0.13 (Edge-connectivity). The *edge-connectivity* of G is
$\kappa'(G) := \max\{\, k : G \text{ is } k\text{-edge-connected} \,\}$.
Equivalently, $\kappa'(G)$ is the size of a smallest edge cut of G.

Definition 0.14 (Edge cut). For a nonempty proper subset $S \subsetneq V(G)$, the *edge cut* determined by S is $[S, \overline{S}] := \{uv \in E(G) : u \in S, v \in \overline{S}\}$.

Definition 0.15 (Bond). A *bond* of G is a minimal edge cut: that is, a set F of edges such that $G - F$ has more components than G, and no proper subset of F has this property.

Deleting any edge cut of a graph G disconnects it, since there will be no path from the two components.

In fact, all minimal disconnecting sets of edges have this form.

Proposition 0.13. Let G be a connected graph.

(i) Every minimal disconnecting set of edges is an edge cut.

(ii) For connected G, an edge cut $[S, \overline{S}]$ is a bond if and only if both induced subgraphs $G[S]$ and $G[\overline{S}]$ are connected.

3.3 Whitney's Inequalities

Theorem 0.14 (Whitney). For every connected graph G, $\kappa(G) \le \kappa'(G) \le \delta(G)$.

Proof. We prove the two inequalities separately.

(i) $\kappa'(G) \le \delta(G)$. Let v be a vertex of minimum degree $\delta(G)$. Deleting all edges incident with v disconnects G, so there exists an edge cut of size at most $\delta(G)$. Hence $\kappa'(G) \le \delta(G)$.

(ii) $\kappa(G) \le \kappa'(G)$. Let $[S, \overline{S}]$ be a minimum edge cut of G, so $|[S, \overline{S}]| = \kappa'(G)$. If every vertex in S is adjacent to every vertex in $\overline{S}$, then $\kappa'(G) = |S||\overline{S}| \ge |S| + |\overline{S}| - 1 = n - 1$. Since $\kappa(G) \le n - 1$, the inequality holds trivially. Otherwise, there exist vertices $x \in S$ and $y \in \overline{S}$ such that $xy \notin E(G)$. We construct a vertex cut separating x and y with size at most $\kappa'(G)$. Define the set T as follows:

$$T = \left(N(x) \cap \overline{S}\right) \cup \left\{z \in S \setminus \{x\} : N(z) \cap \overline{S} \neq \varnothing\right\}.$$

In words, T consists of all neighbors of x in $\overline{S}$ and all vertices in $S \setminus \{x\}$ that are incident to at least one edge of the cut. Note that $x \notin T$ and $y \notin T$ (since $xy \notin E(G)$ and y has no neighbors in $S \cap T = \varnothing$). Since every edge in the cut $[S, \overline{S}]$ is incident to either x (contributing to the first set) or to some $z \in S \setminus \{x\}$ (contributing to the second set), and each vertex in T "accounts for" at least one edge in the cut, we have $|T| \le |[S, \overline{S}]| = \kappa'(G)$.

Finally, we show T separates x and y. Any path from x to y must contain an edge uv with $u \in S$ and $v \in \overline{S}$. If $u = x$, then $v \in N(x) \cap \overline{S} \subseteq T$. If $u \neq x$, then $u \in \{z \in S \setminus \{x\} : N(z) \cap \overline{S} \neq \varnothing\} \subseteq T$. In either case, the path intersects T. Thus $G - T$ is disconnected, so $\kappa(G) \le |T| \le \kappa'(G)$. □

3.4 Further Relations Between Edge Cuts and Degree

When $\kappa'(G) \leq \delta(G)$, no smallest edge cut isolates a vertex.
In fact, both sides of a smallest edge cut must then be larger than the minimum degree of G.
And this follows from a simple expression for the size of an edge cut.

Lemma 0.15. For any subset $S \subseteq V(G)$, $|[S,\overline{S}]| = {}_{v\in S}\, d(v) - 2\,|E(G[S])|$.

Proof. Each edge with both endpoints in S contributes 2 to the sum ${}_{v\in S}\, d(v)$ and does not belong to $[S,\overline{S}]$, while each edge with exactly one endpoint in S contributes 1 to the sum and is counted once in $[S,\overline{S}]$. Edges with both endpoints in $\overline{S}$ contribute neither to the sum over S nor to the cut. □

Lemma 0.16. Let G be a simple graph with minimum degree $\delta(G)$. Suppose $[S,\overline{S}]$ is an edge cut of G with $|[S,\overline{S}]| < \delta(G)$. Then $|S| > \delta(G)$.

Proof. By Lemma 1.5 we know that $\delta(G) > {}_{v\in S}\, d(v) - 2|E(G[S])|$.
But $d(v) \geq \delta(G)$, and $2|E(G[S])| \leq |S|(|S|-1)$.
Therefore, $\delta(G) > |S|\delta(G) - |S|(|S|-1)$, reducing to $|S| > \delta(G)$. □

Theorem 0.17. Let G be a connected simple graph with $\operatorname{diam}(G) = 2$. Then $\kappa'(G) = \delta(G)$.

Proof. By Whitney's inequality, we already know $\kappa'(G) \leq \delta(G)$. It remains to show that $\kappa'(G) \geq \delta(G)$. Assume, for contradiction, that there exists an edge cut $[S,\overline{S}]$ with $|[S,\overline{S}]| \leq \delta(G)$, and among all such cuts choose one of minimum size, so $|[S,\overline{S}]| = \kappa'(G)$. By Lemma 0.16, we have $|S| > \delta(G)$ and $|\overline{S}| > \delta(G)$. If every vertex of S has a neighbor in $\overline{S}$, then each vertex of S contributes at least one edge to the cut, so $|[S,\overline{S}]| \geq |S| > \delta(G)$, contradicting $|[S,\overline{S}]| \leq \delta(G)$.

Hence there exists a vertex $x \in S$ that has no neighbor in $\overline{S}$. By symmetry, there also exists $y \in \overline{S}$ that has no neighbor in S. Any path from x to y must then have length at least 3, since no edge of such a path can go directly between S and $\overline{S}$ from x or to y. This contradicts $\operatorname{diam}(G) = 2$. □

4 k connected graph

4.1 Definitions and Basics

Definition 0.16 (7.2.1)**.** Digraph G is strongly connected if $\forall x, y \in V(G), \exists$ x-y path.

Definition 0.17. It is k-connected if (i) $|V(G)| > k$ (ii) $G - S$ is strong for any $S \subseteq V(G)$ with $|S| < k$.

Definition 0.18. Connectivity $\kappa(G) = \max(k)$, s.t. G is strongly k-connected.

Definition 0.19. For $\emptyset \neq S \subsetneq V(G)$, the edge cut $[S, \bar{S}]$:= set of edges from S to $\bar{S}$.

Definition 0.20. Digraph G is k-edge-connected if every edge cut has size at least k.

Definition 0.21. Edge connectivity $\kappa'(G)$:= minimum size of an edge cut.

4.2 Menger's Theorem (7.2.2)

There are several versions of Menger's theorem. Before we state them we have to do some perparations.

Preparation for the statement:

Definition 0.22. For $xy \notin E(G)$, an (x, y) – separating set is $S \subseteq V(G) - \{x, y\}$ s.t. $V(G) - S$ has no (x, y)-path.

Definition 0.23. $\kappa(x, y) := \min |S|$ of such S.

Definition 0.24. Paths from x to y are independent if they share no internal vertex.

Definition 0.25. $\lambda(x, y) := \max \#$ pairwise independent (x, y)-paths.

Remark 0.6 (Intuition)**.** Observe that $\kappa(x, y) \geq \lambda(x, y)$, since any different path would require different cut-vertex.

Definition 0.26. We generalize the notion of path/barrier/link to subsets of vertices.

- For $X, Y \subseteq V(G)$, X-Y-path is a path from x to y that visits X, Y only at its endpoints.
- strict X-Y-path if only endpoint of the path is in X and Y. (Apparently I have used this interchangeably with X-Y path in the context).
- X-Y cut (barrier) is a vertex set $Z \subseteq V(G)$ s.t. $G - Z$ has no X-Y path. (notice $Z = X$ or $Z = Y$ are such cut).
- X-Y link is a set of pairwise disjoint X-Y paths.

  (Note: pairwise disjoint and independent used interchangeably for vertex-disjoint. Sorry for the confusion).

Statements of Mengers (8 variants including the digraph versions). Now we are ready for the statements.

Theorem 0.18 (Menger 1927, vertex version)**.** $\forall x, y$ with $xy \notin E(G)$, $\kappa(x, y) = \lambda(x, y)$.

Remark 0.7 (Intuition)**.** This is a **min-max type result**, finding min cut is dual with max # independent paths.

Theorem 0.19 (Pym's thm 1969, General Menger's)**.** Let G be a graph (or digraph), and let $X, Y \subseteq V(G)$. Then the minimum size of an X-Y-cut equals the maximum size of an X-Y link.

Proof. Let k be the minimum size of an X-Y cut. We proceed by induction on the number of edges in G, denoted by $m = |E(G)|$. Base case: If $m = 0$, then all paths are single vertices (if $X \cap Y \neq \emptyset$) or no paths exist. The result holds trivially.

Case 1: There exists a minimum X-Y cut Z such that $Z \neq X$ and $Z \neq Y$.

Let G_1 be the subgraph consisting of all vertices and edges reachable from X along paths that end at Z.

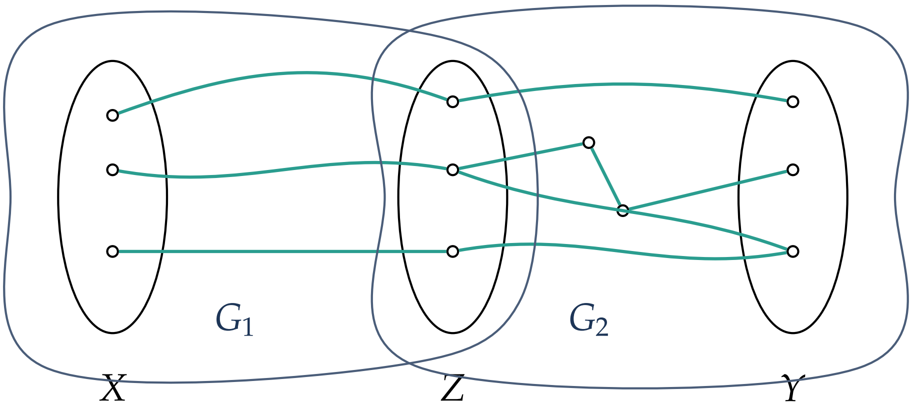
Let G_2 be the subgraph consisting of all vertices and edges reachable from Z along paths that end at Y.

Since Z is an X-Y cut in G, any X-Y path in G must pass through Z. Thus, every X-Y path is a concatenation of an X-Z path in G_1 and a Z-Y path in G_2.

Furthermore, since $Z \neq X$ and $Z \neq Y$, both G_1 and G_2 have strictly fewer edges than G (we can assume minimal graphs where edges are relevant). By the Inductive Hypothesis applied to G_1, there exists a system of k vertex-disjoint X-Z paths.

By the Inductive Hypothesis applied to G_2, there exists a system of k vertex-disjoint Z-Y paths.

These two sets of paths can be concatenated at the vertices of Z to form k vertex-disjoint X-Y paths in G.



Case 2: Every minimum X-Y cut is either X or Y.

Without loss of generality, assume $|X| = k$ and $X \neq Y$. (If $X \subseteq Y$, the shortest paths are length 0, and we trivially have k paths). Since $X \neq Y$, there exists an edge $e = uv$ reachable from X that is not entirely within Y.

Consider the graph $G' = G - e$. Let k' be the minimum size of an X-Y cut in G'.

If $k' = k$, then by the Inductive Hypothesis, there exist k vertex-disjoint paths in G', which are also valid in G.

If $k' < k$, then there exists an X-Y cut Z' in G' of size $k - 1$.

In the original graph G, Z' is not a cut, so the edge e must bridge the separation caused by Z'.

This implies that $Z' \cup \{u\}$ and $Z' \cup \{v\}$ are X-Y cuts in G.

Since $|Z'| = k - 1$, both $|Z' \cup \{u\}|$ and $|Z' \cup \{v\}|$ are at most k.

Since the minimum cut size in G is k, both must have size exactly k.

By the assumption of Case 2, any minimum cut in G must be X or Y.

Thus, $\{Z' \cup \{u\}, Z' \cup \{v\}\} \subseteq \{X, Y\}$.

This forces a specific structure (e.g., $Z' \cup \{u\} = X$) allowing us to construct the paths directly: we take the paths avoiding Z' (from the cut property) and the path through e.

Specifically, since $Z' \subset X$ and $|Z'| = |X| - 1$, the vertices of X are exactly $Z' \cup \{u\}$.

Similarly if $Z' \cup \{v\} = Y$. We can pair the vertices of X to Y directly to form the link. $\square$

Remark 0.8 (Note)**.** Observe that the standard vertex version of Menger's Theorem follows from Pym's Theorem by setting $G' := G - \{x, y\}$, $X := N(x)$, and $Y := N(y)$.

4.3 Menger's edge version

Definition 0.27. We need some definitions in order to state the edge version of Menger's theorem.

- $\kappa'(x, y) :=$ min # edges whose removal makes y unreachable from x.
- $\lambda'(x, y) :=$ max # pairwise edge-disjoint paths.
- Line graph
  $L(G)$ of G: $V(L(G)) := E(G), E(L(G)) := \{ef : e, f \in E(G), \text{which share endpoints}\}$.
- For digraphs the edge set is ordered: $E(L(G)) := \{e \to f : e, f \in E(G), e = u \to v, f = v \to z\}$.

Remark 0.9 (Note)**.** Note that $\alpha'(G) = \alpha(L(G))$.

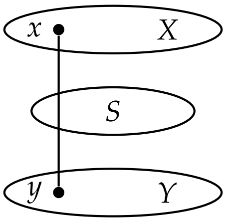
Theorem 0.20 (Menger's theorem edge version)**.** For graph/digraph G, $\kappa'(x,y) = \lambda'(x,y)$ $\forall x,y \in V(G)$.

Proof. $X := N(x)$, $Y := N(y)$. Apply Pym's thm, and remove $(x,y) \implies \kappa(x,y) = \lambda(x,y)$. Apply Pym's thm to $L(G)$, where $X :=$ edges leaving x, $Y :=$ edges leaving y. Then vertex disjoint path in $L(G) \equiv$ edge disjoint path in G. □

Next we intend to state the global version of Menger's theorem, which was observed by Whitney. The global edge version was observed by Ford-Fulkerson, and for the vertices version we need the following lemma.

Lemma 0.21 (7.2.7)**.** Deletion of an edge reduces connectivity by at most 1.

Proof. If S is a smallest separating set of $G - xy$, then $S \cup \{x\}$ or $S \cup \{y\}$ separates G (yielding $\kappa(G) \leq \kappa(G - xy) + 1$), unless x and y are the only vertices of $G - S$. In that case $|S| = |V(G)| - 2 \geq \kappa(G) - 1$, so again $\kappa(G) \leq \kappa(G - xy) + 1$.



□

Theorem 0.22 (7.2.8 Menger-Whitney)**.** For a graph or digraph G,

(i) G is k-connected iff $\lambda(x,y) \geq k$ for all $x,y \in V(G)$.

(ii) G is k-edge-connected iff $\lambda'(x,y) \geq k$ for all $x,y \in V(G)$.

Proof. Since $\kappa'(G) = \min_{x,y \in V(G)} \kappa'(x,y)$, the Menger edge theorem yields (ii).

For $\kappa(G)$, the Menger vertex theorem yields $\kappa(x,y) = \lambda(x,y)$ for $xy \notin E(G)$, and $\kappa(G)$ is the least of these values. We need only show that $\lambda(x,y) \geq \kappa(G)$ for $xy \in E(G)$. Since xy forms an x,y-path and lies in no other x,y-path:

$$\lambda_G(x,y) = 1 + \lambda_{G-xy}(x,y) = 1 + \kappa_{G-xy}(x,y) \geq 1 + \kappa(G - xy) \geq \kappa(G)$$

where the first inequality comes from the definition of global connectivity relative to local connectivity, and the second inequality comes from Lemma 7.2.7. □

5 Application of Menger's theorem

Corollary 0.23 (7.2.9)**.** $\kappa(G) = \kappa'(G)$ when $\Delta(G) \leq 3$.

Proof. $\kappa(G) \leq \kappa'(G)$: remove one endpoint removes an edge. $\forall x, y \in V(G)$, WTS $\exists\ \kappa'(G)$ vertex-disjoint (independent) paths. But we know $\exists\ \kappa'(G)$ edge-disjoint paths. If a vertex is to be reused by two paths, its degree is at least 4, a contradiction. ☐

Lemma 0.24 (7.2.10 Expansion Lemma)**.** If G' is formed from a k-connected graph G by adding a vertex y with at least k neighbors in G, then G' is k-connected.

Proof. $|S| < k$. $G - S$ is connected by definition. But then the deletion always remains at least one neighbor of y in the graph. Thus $G' - S$ is connected. ☐

Definition 0.28 (7.2.11)**.** For $x \in V(G)$, $U \subseteq V(G)$, an $\underline{x, U\text{-fan}}$ of size k is a set of k paths from x to U that pairwise only share x and reach U only at endpoints.

Lemma 0.25 (7.2.12 Fan Lemma, Dirac)**.** G with more than k vertices is k-connected iff $\forall x \in V(G), U \subseteq V(G), |U| \geq k, x \notin U$, there exists an x, U-fan.

Proof. ($\Rightarrow$): Construct $G' := G + y$, y adjacent to all of U. By expansion lemma, G' is k-connected. By Menger, $\forall x$, there exists a set of k independent paths. Early stopping the paths when they reach U gives an x, U-fan.

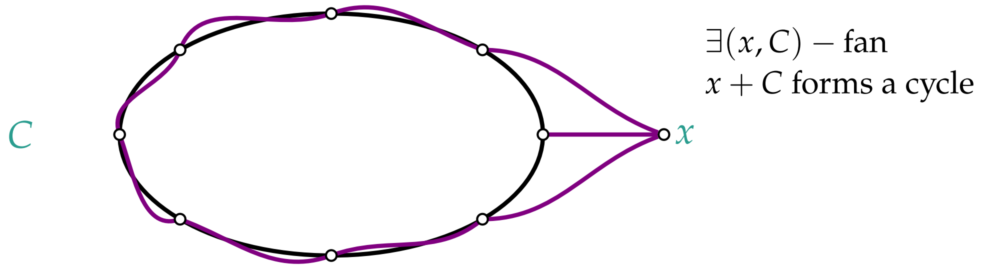
($\Leftarrow$): Prove the contrapositive. Since G is not k-connected, it has a vertex cut of size $(k-1)$. Choose x, y in different component of $G - S$. take $U := S \cup \{y\}$. But then any x-y path would use a vertex on S, thus there is no x-U-fan. ☐

Theorem 0.26 (7.2.13 Dirac 1960)**.** Let $k \geq 2$. In any k-connected graph G, each set of k vertices lies on some cycle.

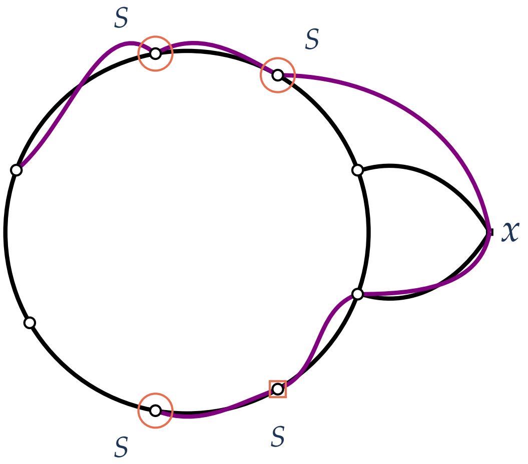
Proof. Induction on k. Base $k = 2$: $\forall x, y \in V(G)$, by Mengers $\exists$ two independent paths from x to y, of which forming a cycle containing x, y.

Remark 0.10 (Strategy)**.** Induction Step: $k \geq 3$. Let $|S| = k$, $x \in S$. Since G is $(k-1)$-connected, apply IH, $S - \{x\}$ lies on some cycle C.

(i) If $|V(C)| = k - 1$, consider $x, V(C)$-fan of size $k - 1$ (possible by fan lemma). Find 2 paths from x to $V(C)$, and concatenate the rest of the cycle C we obtain the desired cycle.



(ii) If $|V(C)| \geq k$ and $x \notin V(C)$, the above argument doesn't apply, since the constructed cycle contain vertices not in S, and paths from x to $S - x$ might intersect $C - S$, making the construction of the cycle illegal. Consider that $S - x$ partitions C into $k - 1$ intervals. Find an $x, V(C)$-fan, and each partition starts at an element of $S - \{x\}$. Since the fan has size k, $\exists$ two paths reach C within the same segment. Then detour from those two vertices to visit x, and we have obtained the desired cycle.



□

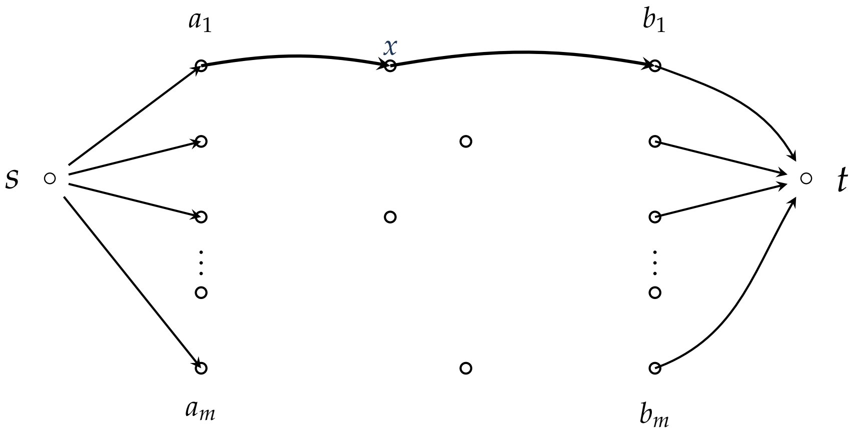
Ford-Fulkerson SDR

Definition 0.29. (System of Distinct Representative) SDR of sets $A_1, \ldots, A_m$ is distinct $x_1, \ldots, x_m$ s.t. $x_i \in A_i$. (Common SDR) CSDR is an SDR for both A and B, A, B two families of sets each of size m.

Theorem 0.27 (7.2.15 Ford-Fulkerson 1958)**.** Let A, B 2 families of m sets: $A = \{A_1, \ldots, A_m\}$, $B = \{B_1, \ldots, B_m\}$. $A(I) = {}_{i \in I} A_i$, $B(J) = {}_{j \in J} B_j$. A, B have a CSDR iff $|A(I) \cap B(J)| \geq |I| + |J| - m$ for each $I, J \in [m]$.

Proof. Prove the equivalent: $|A(I) \cap B(J)| + (m - |I|) + (m - |J|) \geq m$ for all $(I), (J)$ (*).

Remark 0.11 (Explanation)**.** Construct digraph G with s, t, vertex sets A', B', X, where $A' = \{a_i : A_i \in A\}$, $B' = \{b_j : B_j \in B\}$, $X = (\ A_i) \cup (\ B_j)$. Edges: $\{sa_i\} \cup \{a_i x : x \in A_i\} \cup \{b_j t\} \cup \{xb_j : x \in B_j\}$. An s-t path is of the form $< s, a_i, x, b_j, t >$ for $x \in A_i \cap B_j$. Thus $\exists$ CSDR $\iff$ $\exists\ m$ independent s-t paths.



$\overset{\text{Menger}}{\Longleftrightarrow}$ every s-t separating set has size at least m. Let R be minimal s-t separating set, and $I := \{i \in [m] : a_i \notin R\}$, $J := \{j \in [m] : b_j \notin R\}$. The vertices of A', B' indexed by I and J are not being deleted in $G - R$. R is an s-t cut iff $A(I) \cap B(J) \subseteq R$. Therefore

$$|R| = |A(I) \cap B(J)| + (m - |I|) + (m - |J|)$$

Thus $|R| \geq m \iff \exists$ CSDR for A, B, we are done. □

6 Hamilton (Spanning) cycle

Definition 0.30. A vertex spanning cycle on G is called a hamilton cycle. The graph containing it is called hamiltonian.

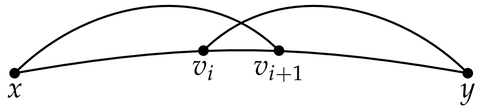
Proposition 0.28 (7.3.2)**.** If G has Hamilton cycle, then $[\#\text{components}(G - S) \leq |S|], \forall \emptyset \neq S \subseteq V(G)$.

Proof. Whenever leaving $G - S$ for S, arrivals have to be distinct, thus S has at least as many vertices as $G - S$ has components. □

Remark 0.12 (Note)**.** This is a necessary condition.

Lemma 0.29 (7.3.6 Ore's Lemma)**.** Let $x, y \in V(G), xy \notin E(G), |V(G)| = n$. If $d(x) + d(y) \geq n$, then G has H-cycle $\iff$ $G + \{xy\}$ has H-cycle.

Proof. G H-cycle $\implies$ $G + \{xy\}$ H-cycle. ($\Leftarrow$): If $G + \{xy\}$ H-cycle but G not, then xy is in all H-cycle of $G + \{xy\}$. Index vertices $v_1, \dots, v_n$ along one spanning x, y-path in G. Consider the case that some neighbor of x immediately follows a neighbor of y: Then the H-cycle can be constructed by $[x, v_{i+1}, \dots, y, v_i, x]$.



We show that such case always show up by proving S, T have common element, $S := \{i : v_{i+1} \in N(x)\}$, $T := \{i : v_i \in N(y)\}$. $|S \cup T| + |S \cap T| = |S| + |T| = d(x) + d(y) \geq n$. But neither contain index n. thus $|S \cup T| < n$, hence $|S \cap T| \geq 1$. $\implies$ conclude G has a H-cycle. □

Generalizing this local Ore's lemma, we obtain the following sufficient condition for Hamilton cycle.

Corollary 0.30 (7.3.7 Ore's thm)**.** $\forall n \geq 3$, G n-vertex graph. $\forall x, y, xy \notin E(G), d(x) + d(y) \geq n$. Then G has a H-cycle.

Proof. Assume for contradiction G has no H-cycle but satisfies the condition. $xy \notin E(G)$, $d(x) + d(y) \geq n$. By Ore's lemma keep adding edges doesn't create Hamilton cycle, but K_n has Hamilton cycle, which is a contradiction. □

7 Vertex coloring

Definition 0.31 (8.1.1)**.** A k-coloring of G is a vertex labeling $f : V(G) \to S$, $|S| = k$. The coloring f is proper if $f(x) \neq f(y)$ whenever $xy \in E(G)$, and G is k-colorable if it has a proper k-coloring.

The chromatic number $\chi(G)$ is the least k s.t. G is k-colorable. If $\chi(G) = k$, G is k-chromatic.

Remark 0.13 (Note)**.** k-colorable $\iff$ if and only if k-partite.

Example 0.1. 2-colorable $\iff$ bipartite $\iff$ no odd cycles.

Definition 0.32. $\alpha(G)$ is max independence number, $\omega(G)$ is the maximum size for clique, called clique number.

Remark 0.14. We state some basic facts about chromatic number.

- $\chi(G) \leq n$.
- $\chi(G) \geq \frac{n}{\alpha(G)}$, since each color class is an independent set.
- $\chi(G) \geq \omega(G)$, since adjacent vertices require different colors.
- $H \subseteq G \implies \chi(H) \leq \chi(G)$.
- $\omega(G) = 2$, $\chi(G)$ can be infinitely large.

7.1 Greedy Coloring

8.1.5. Greedy coloring Principle: For graph G, a ordering of vertices $v_1, v_2, \ldots, v_n$. Color the vertices in the given order, while giving v_i the least indexed color not used on its neighbors colored earlier.

Proposition 0.31 (Claim 1.6)**.** $\chi(G) \leq \Delta(G) + 1$.

Proof. Each vertex has at most $\Delta(G)$ earlier neighbors. Thus for any ordering, at least one color is available. □

Theorem 0.32 (1.7 Brook's theorem)**.** $\chi(G) = \Delta(G) + 1$ only if G has $K_{\Delta(G)+1}$ or an odd cycle when $\Delta(G) = 2$. (sharp bound).

Proposition 0.33 (1.10)**.** This is a natural bound based on degree sequence. We will reuse this idea in the k-generate proof.

Graph G has vertex degrees $d_1, \ldots, d_n$, then $\chi(G) \leq 1 + \max_i \min\{d_i, i - 1\}$.

Proof. When coloring vertex i, the number of neighbors already colored is bounded by d_i or $i-1$. You need an additional color to color vertex i. Iterating through each vertex and take the maximum needed. □

Definition 0.33 (1.11). G is k-degenerate if every subgraph has a vertex of degree at most k. The degeneracy of graph G is $\max_{H\subseteq G}\delta(H)$, i.e., the minimum k s.t. G is k-degenerate.

- 1-degenerate $\iff$ acyclic graph.

Proposition 0.34 (1.12 Szekeres-Wilf bound). If G is k-degenerate, then
(i) G is $(k+1)$-colorable.
(ii) In particular, $\chi(G) \leq 1 + \max_{H\subseteq G}\delta(H)$.

Proof. (i): For G, construct a smallest-last ordering, from n to 1, and v_i be the vertex of minimum degree among $G - \{v_{i+1}, \dots, v_n\}$.

Remark 0.15 (Intuition). This ordering utilizes the property of k-degenerate. Every time you pick the smallest vertex, it has degree smaller than k in the remaining graph, which is the vertices before the current vertex.

Then this ordering gives at most k neighbors among earlier vertices. Hence the greedy coloring for such a vertex ordering is a proper $(k+1)$-coloring.

(ii) the maximum of $\delta(H)$ is the least k s.t. G is k-degenerate.

□

Theorem 0.35 (1.14 Gallai-Roy Theorem, statement only). $\chi(G) \leq 1 + l(D)$ for every orientation D of G, where $l(D)$ is the length of a longest path in D.

1.15 Mycielski's construction: Given graph G, construct G'. $V(G') = \{v_1, \dots, v_n\}$. Add vertex w, an independent set $U = \{u_1, \dots, u_n\}$. For each i, let u_i adjacent to all of $N_G(v_i)$. Finally let $N_{G'}(w) = U$.

Theorem 0.36 (1.17). (Mycielski's construction gives arbitrarily large triangle-free graphs.) Given G, construct G' through Mycielski's construction. If G is triangle free G' also triangle free. If $\chi(G) = k$, $\chi(G') = k+1$.

Proof. (i) Assume G is triangle free. Triangles in G' must contain vertices in U, and are neighbors of v_i. But then since u_i and v_i have the same copy of neighbors, this produces a triangle consisting only of vertices in G, contradiction. Thus G' does not contain triangle.

(ii) We claim a proper $(k+1)$-coloring of G'. Define $f(u_i) = f(v_i)$, $f(w) = k+1$. This coloring is proper since u_i preserves adjacency from v_i, and w is adjacent to all of u_i. This establishes the upper bound, i.e., $\chi(G') \leq \chi(G) + 1$.

Next we show $\chi(G) < \chi(G')$

Remark 0.16 (Strategy). Proof sketch: show that G' k-colorable $\implies$ G $(k-1)$-colorable; thus coloring is increasing and establish the lower bound.

Let's assume that G' has proper k-coloring g, WLOG assume $g(w) = k$. Then g is restricted to $[k-1]$ on U. We then proceed to construct the desired $(k-1)$-coloring. Let $A = \{v_i : g(v_i) = k\}$, i.e., vertices in A that are colored k. (If we can recolor these vertices from colors within $[k-1]$ we are done.) Indeed, we recolor vertices in A by $g(u_i)$. We claim this yields a proper $(k-1)$-coloring of G.

$G - A$ is already proper colored. A is independent since it's a color class. Left to check colors on $\{v_i, v'\}$ with $v_i \in A$ and $v' \in V(G) - A$. If $v_iv' \in E(G)$, then $u_iv' \in E(G')$ by construction. This means $g(v') \neq g(u_i)$. But then v_i also has different colors than its neighbors in $V(G) - A$. Finally we delete U and $\{w\}$ to obtain the proper $(k-1)$-coloring of G. □

Proposition 0.37 (18)**.** If G is an n-vertex triangle-free graph, then $\chi(G) \leq 2\sqrt{n}$. Equivalently, $\alpha(G) \geq \frac{\sqrt{n}}{2}$ if G triangle free and $\chi(G) = k$.

Proof. We analyze two cases.

(i) If $\exists v$ with $d(v) \geq \sqrt{n}$, then color $N(v)$ with same color.

(ii) If no such vertex exists, the subgraph induced by the remaining vertices has maximum degree less than $\lfloor \sqrt{n} \rfloor$. Using greedy coloring we can color the remaining graph with at most $\sqrt{n}$ colors. The first step uses at most $\frac{n}{\sqrt{n}} = \sqrt{n}$ colors. In total we use at most $2\sqrt{n}$ colors. □

8 Color-critical graph

Definition 0.34 (8.2.1). Graph G is color-critical if $\chi(H) < \chi(G)$ for every proper subgraph H of G. In particular, if $\chi(G) = k$, we call G k-critical.

Remark 0.17 (Intuition). removal of arbitrary edge drop chromatic number.

- 1-critical graphs is K_1.
- 2-critical graph is K_2.
- 3-critical graph are odd cycles.

Proposition 0.38 (2.2). G is k-critical, then $\delta(G) \geq k - 1$.

Proof. Assume x has smaller degrees. $G - x$ has a $(k-1)$-coloring, guaranteed by k-criticalness. But then by giving x a color not used in $N(x)$ we get a $(k-1)$-coloring of G, contradiction. □

Remark 0.18 (Strategy). Assume failure on k-critical graph G. Then obtain a proper $(k-1)$-coloring of an appropriate subgraph of G. Then use the failure to extend a proper $(k-1)$-coloring to G which gives a contradiction.

Remark 0.19. Below we give some characterization of k-critical graph.

(i) $\chi(G - v) < \chi(G) = k \implies$ G has a proper k-coloring s.t. v has a unique color and all other colors appear in $N(v)$.

(ii) $\chi(G - e) < \chi(G) = k \implies$ every $(k-1)$-coloring of $G - e$ assigns same color to the endpoints of e.

Proof. (* practise the intuition above.)

(i) Let f be $(k-1)$-coloring of $G - v$. Assume any color is not used in $N(v)$, then use it to color v will give rise to a $(k-1)$-coloring of G.

(ii) Let f be $(k-1)$-coloring on $G - e$ but giving distinct colors to endpoints of e. But then it's a proper $(k-1)$-coloring of G. □

Theorem 0.39 (2.5 Dirac). Every k-critical graph is $(k-1)$-edge-connected.

Proof. Let G be k-critical. Let $[X, Y]$ be an edge cut. Since G is k-critical, subgraph $G[X]$ and $G[Y]$ are $(k-1)$-colorable.

Let $X_1, \ldots, X_{k-1}$ and $Y_1, \ldots, Y_{k-1}$ be the color classes in the proper $(k-1)$-coloring of $G[X]$ and $G[Y]$.

We want to show that $|[X, Y]| \geq k - 1$. We show this by contradiction based on Induction.

First use induction to show that under this $(k-1)$-coloring of X and Y, if $|[X, Y]| < k - 1$, then we can reindex Y_i s.t. $[X_i, Y_i]$ is empty for all i. But then the separate $(k-1)$-coloring for $G[X]$ and $G[Y]$ give rise to a $(k-1)$-coloring proper to G, since you can reuse the same color for corresponding color classes in Y.

But then this is a contradiction since G is k-critical.

Finally we finish the induction steps.

Base case: $|[X, Y]| < 1$, the set is empty so trivial.

Induction hypothesis: $|[X, Y]| < r$ with partitions $X_1 \dots X_r$, $Y_1 \dots Y_r$ can be reindexed s.t. $|[X_i, Y_i]| = 0$ for all $k < r$. Some X_i is incident to no edges of $[X, Y]$ (Because pigeonhole).

If the set $[X, Y]$ is nonempty, pick edge $e \in [X, Y]$, and Y_i be incident to e.

For the pairing, we pair X_i with Y_i, which $[X_i, Y_i]$ has no edges.

Then we can remove X_i and Y_i, reducing the graph to G'.

Induction hypothesis applies since we have at most $r-1$ sets in X and Y, and we delete no edges.

The pairing in G' with (X_i, Y_i) gives us the desired pairing. □

8.1 List coloring

Definition 0.35 (8.2.6)**.** For graph G, a list assignment L assigns to each vertex $v \in V(G)$ a set $L(v)$ of colors allowed at v. a L-coloring is a proper coloring ϕ of G s.t. $\phi(v) \in L(v)$ for all v. G is k-choosable / list k-colorable if it has an L-coloring whenever $|L(v)| \geq k$ for all v. The list chromatic number / choice number / choosability $\chi_L(G)$ is the minimum k s.t. G is k-choosable.

Remark 0.20 (Note)**.** lists are set. $\chi_L(G) \geq \chi(G)$, equality achieved when $|S| = \chi(G)$.

- cycles are 2-choosable.
- $\chi_L(K_{m,m}) > k$ when $m = \binom{2k-1}{k}$.

Remark 0.21 (Explanation)**.** Explanation for $K_{m,m}$

Let L assign all k-subsets of $[2k-1]$ as lists. In an L-coloring, at least k colors must be used on each part, otherwise some vertex has no color chosen from the k in its list. But the lists have $2k-1$ colors in total, thus some color is used on both part. But $K_{m,m}$ is complete bipartite, same color on both sides means the coloring is not proper. Thus $K_{m,m}$ not L-colorable for these lists of size k, it's not k-choosable.

A similar result from vertex coloring. Proof follows similarly.

Proposition 0.40 (8.2.8 Szekeres-Wilf bound)**.** Every k-degenerate graph is $(k+1)$-choosable. Thus also $\chi_L(G) \leq 1 + \max_{H \subseteq G} \delta(H) \leq 1 + \Delta(G)$. Note that this is improved from the vertex version.

Proof. Proof is similar. Every time delete a vertex with degree at most k. Now in the reverse order of this deletion, each vertex has at most k earlier neighbors. Given all lists have size at least $k+1$, base on a similar argument from 8.1.12/8.1.6, it always has a color not used. □

8.2.14 (Brook's theorem list extension), statement only. If a connected graph G is not a complete graph or odd cycle, then $\chi_L(G) \leq \Delta(G)$.

9 Edge coloring

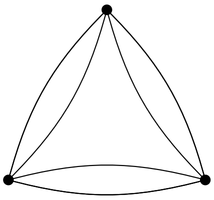
Definition 0.36 (8.3.1). a k-edge-coloring of G is a labeling of its edges from a set of k colors. The coloring is proper if incident edges have distinct colors. G is k-edge-colorable is a proper k-edge-coloring exists. The edge chromatic number / chromatic index $\chi'(G)$ of G is minimum k s.t. G is k-edge-colorable.

Remark 0.22 (Note). We state some basic facts about edge chromatic number.
$\chi'(G) = \chi(L(G))$
$\chi'(G) \geq \Delta(G)$
$\chi'(G) \leq 2\Delta(G) - 1$ ($\Delta(L(G)) \leq 2\Delta(G) - 2$, (consider edge uv in G, and assuming both endpoints have maximum degree, then bound the degree in its line graph.)

Remark 0.23 (Vizing and Gupta). gave the bound $\chi'(G) \leq \Delta(G) + \mu(G)$, $\mu(G)$ the max edge multiplicity. In a graph G this reduces to $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Definition 0.37 (8.3.4). A graph G is class 1 when $\chi'(G) = \Delta(G)$. class 2 otherwise. A decomposition of a multigraph into perfect matching is a 1-factorization.

- Fat triangle (Shannon)



$\chi'(G) = \frac{3}{2}\Delta(G)$

Observe: A regular graph is class 1 iff it has a 1-factorization.

Remark 0.24 (Explanation). Reasoning of why Petersen graph is class 2

For petersen graph P, $\Delta(P) = 3$, so $\chi'(P) = 3$ or 4. Assume it's 3-edge-colorable. Since it's 3-regular, each vertex incident to one edge from each color. Thus the edges of each color class form a perfect matching M. Deleting M from P, $P - M$ is 2-regular, disjoint union of cycles. Since it's 2-colored, the cycles are alternating, thus they are even cycles. But petersen has girth 5, $|V(P)| = 10$. Thus either 10 or $5 + 5$. But P is not hamiltonian, thus $P - M$ decompose into $2C_5$, a contradiction. Therefore $\chi'(P) = 4$, P is class 2 graph.

Theorem 0.41 (8.3.7 Konig 1916). $\chi'(G) = \Delta(G)$ when G is a bipartite multigraph.

Proof. Construct supergraph H of G, which is a k-regular bipartite graph, $k = \Delta(G)$. The method is to first adding vertices to the smaller of X and Y untill equal, then add edges joining X, Y with degree less than k untill all has degree k. H is k-regular, by marriage theorem it has a 1-factor. Removing the 1-factor, it's $(k-1)$ regular. Inductively decompose this bipartite into k 1-factors, thus you need at most k colors for H, $\chi'(H) \leq k$. But $\chi'(H) \geq \Delta(H) = k \implies \chi'(H) = k$. But $\chi'(G) \leq \chi'(H) = k$, $\chi'(G) \geq \Delta(G) = k \implies \chi'(G) = k$. □

10 Planar graph

10.1 Definitions

Definition 0.38 (9.1.2). We first state some basic facts about planar graphs. The definitions here follows from intuitive understanding of the concept.

- Drawing of a graph maps vertices into points, edges into curves, continuous.
- A crossing is a common internal point. No 3 edges share internal point, no edge has vertex as internal point, no edges are tangent.
- Embedding is a drawing without crossing.
- Graph G is planar if it has a non-crossing embedding in the plane. The actual drawing is the plane graph.
- Faces of a graph is the maximal regions on the plane disjoint from the edges.
- Length of face := number of edges with multiplicity.

Remark 0.25. For rigor, graphs here should be extended to multigraph.

Definition 0.39 (Dual Graph). The dual G^* of a planar G is graph s.t. $V(G^*) :=$ faces of G, and there is a bijection between the edges of G^* and G, $E(G^*) \leftrightarrow E(G)$ in which it joins vertices corresponding to the faces of G whose boundary includes $E(G)$.

Remark 0.26. $(G^*)^* \cong G$ iff G is connected. Cut edges are loops.

Lemma 0.42 (9.1.10 Handshake/Dual degree sum). $l(f_i)$ is the length of face f_i, if G has m edges, $l(f_i) = 2m$.

Proof. Apply degree sum formula on G^*, since $l(f_i)$ is the degree of f_i in G^*. (Also intuitively, every edge is counted twice in both side.) □

Proposition 0.43 (9.1.11). Edges in a plane graph form a cycle iff dual edges form a bond in G^*.

Theorem 0.44 (9.1.14)**.** G is a plane graph. Then the following are equivalent:

(A) G is bipartite.

(B) Every face of G has even length.

(C) Dual G^* is Eulerian.

Proof. $A \Rightarrow B$: Faces are cycles. Bipartite graphs only have even cycles.

$B \iff C$: dual G^* is connected. Degrees in G^* are face lengths in G, thus G^* has even degree which imply Eulerian.

$B \Rightarrow A$: Let C be a cycle in G. Every face of G is either wholly inside of C or outside of C. Summing face length inside C is even since each face is even length, but the sum counts each edge of C once, and counts each edge inside C twice. Hence the length of C is even. All cycles even $\iff$ G is bipartite. □

10.2 Outerplanar Graphs

Definition 0.40 (9.1.15)**.** Graph G is outerplanar if it has an embedding with every vertex on the boundary of the unbounded face.

Proposition 0.45 (9.1.16)**.** The boundary of the outer face of a 2-connected outerplane graph is a spanning cycle.

Proof. Traversing the boundary of the outerface includes all vertices. If not a cycle it includes some vertices more than once, then it's a cut vertex, contradiction. □

Example 0.2. $K_4, K_{2,3}$ are planar, not outerplanar. $K_{2,3}$ is 2-connected but has no spanning cycle. K_4 spanning cycle requires the endpoint of the remaining 2 edges alternate which can both be drawn inside.

Proposition 0.46 (9.1.20)**.** Every simple outerplanar graph has a vertex of degree at most 2.

10.3 Euler's Formula

Theorem 0.47 (9.1.21 Euler Formula)**.** If a connected plane multigraph G has n vertices, m edges, f faces, then

$$n - m + f = 2$$

Proof. Induction on n. Base $n = 1$ is a bunch of loops. The formula $f - m = 1$ holds because each edge creates an inner face and they all share an outerface.

Remark 0.27 (Strategy)**.** Induction: Contracting a non-loop edge in G, with parameters n', m', f'.

Note $n' = n - 1$, $m' = m - 1$. But $f' = f$ since we just shortened the boundary. By induction hypothesis, $n' - m' + f' = n - 1 - (m - 1) + f = n - m + f = 2$. □

Remark 0.28 (Aside)**.** # faces are same despite different embeddings. For unconnected graphs, $n - m + f = k + 1$, k the number of components.

Theorem 0.48 (9.1.23). G simple, n-vertex, planar, m edges. If $n \geq 3$, then $m \leq 3n - 6$. If G is triangle free, $m \leq 2n - 4$.

Remark 0.29. With 9.1.23, we can see K_5 is not planar since it has 10 edges ($10 \not\leq 3(5) - 6$). Same for $K_{3,3}$.

Proof. We first add edges to make G connected. For $n \geq 3$, every face boundary in a connected plane graph has length at least 3 (Since G is simple). We also know that $_{i \geq 3} l(f_i) = 2m$, therefore $2m \geq 3f$, or $f \leq \frac{2m}{3}$. Substituting into Euler's formula: $2 = n - m + f \leq n - m + \frac{2m}{3} = n - \frac{m}{3}$. This gives $m \leq 3n - 6$.

When there's no triangle, we can improve the bound for faces to at least length of 4. Thus $2m \geq 4f$, $f \leq \frac{m}{2}$. This gives $m \leq 2n - 4$. □

10.4 Maximal Planar Graphs

Definition 0.41 (9.1.25). A maximal planar graph is a planar graph that is not a spanning subgraph of another planar graph. A triangulation is a plane multigraph where every face boundary is a triangle.

Proposition 0.49 (9.1.26). For n-vertex plane graph, following are equivalent:

(a) G is maximal planar.

(b) G has $3n - 6$ edges.

(c) G is a triangulation.

Proof. $b \iff c$: According to 9.1.23, having $3n - 6$ edges means $2m = 3f$. This is true only if all faces are triangle.

$a \iff c$: If some faces have length more than 3, there is always a way to add edges to create a larger planar graph. Thus G is max planar iff all faces are triangle. □

Remark 0.30 (9.1.28). $\exists$ embedding in a plane $\iff$ $\exists$ embedding in a sphere. (Imagine puncture a face on the sphere, then extend it to the infinite face on the plane.)

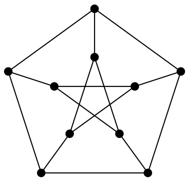
11 Structure of planar graph

Definition 0.42 (9.2.1). A subdivision of an edge is adding new vertices inside existing edges. The subdivision of a graph is graph obtained through edge subdivisions.

Theorem 0.50 (9.2.3 Kuratowski). Graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. (Notice this means the subdivisions "of" K_5 or $K_{3,3}$).

Proof. ($\Leftarrow$): If it contains it, subdivision preserves planarity, and $K_5, K_{3,3}$ are not planar, thus the graph is not planar. The other direction. ($\Rightarrow$): If it's planar, then the subgraph of the graph is planar. Also every subdivision of a nonplanar graph is nonplanar. Thus the theorem follows. □

- **Theorem (Wagner):** G is planar if and only if it has no subgraphs contractible to K_5 or $K_{3,3}$.
- **Theorem (Fary):** G is planar iff it has a drawing of only straight segment.
- **Theorem (Tutte):** G is 3-connected, with no subdivision of $K_{3,3}$ or K_5, then G has a convex embedding.



Remark 0.31 (Example: Petersen is not planar)**.** Can't contain subdivision of $K_5 \to$ contains sub of $K_{3,3}$? Girth 5 $\implies 2m \geq 5f$. But $f = 2 - n + m$. $\implies 2m \geq 10 - 5n + 5m$. $\implies m \leq \frac{5}{3}(n-2) \implies m \leq 13.33$. But Petersen has 15 edges, contradiction.

12 Ramsey's Theorem

Remark 0.32 (Intuition). For a sufficiently large graph, there exists a Ramsey number $R(k,l)$ such that no matter how you color the edges, you get a monochromatic complete graph of either size k (red) or size l (blue).

- Example: $R(3,3) = 6$. If you color the edges of K_6 red/blue, you are guaranteed a blue triangle or a red triangle.

More formally, we define:

Definition 0.43 (10.2.2). A k-coloring is a function that labels each domain element with one of k colors. Let $\binom{S}{r}$ denote the family of r-sets of a set S. When coloring $\binom{S}{r}$, a set $T \subseteq S$ is homogeneous if its r-sets all have the same color.

Remark 0.33. An "edge-coloring" is simply the case when $r = 2$. Compare this general set coloring strategy to the edge-coloring of hypergraphs.

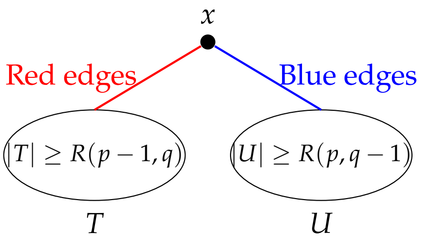
Definition 0.44 (10.2.3). In a k-coloring of $\binom{S}{r}$, a homogeneous set with color i is i-homogeneous.

Given quotas $p_1, \ldots, p_k \in \mathcal{N}$, the Ramsey number $R(p_1, \ldots, p_k; r)$ is the least N s.t. every k-coloring of $\binom{[N]}{r}$ has an i-homogeneous set of size p_i for some i.

12.1 Graph Ramsey (r=2)

Proposition 0.51. $R(p,q) \le R(p-1,q) + R(p,q-1)$.

Proof. Let $N = R(p-1,q) + R(p,q-1)$. Choose a vertex x. Pigeonhole Principle implies that since there are $N-1$ vertices other than x, x has either N_{red} incident red edges or N_{blue} incident blue edges, where $N_{red} = R(p-1,q)$ and $N_{blue} = R(p,q-1)$.



Let T be the set of neighbors connected to x by red edges, and U be those by blue edges. Consider a 2-coloring of E_N. Assume x has at least N_{red} incident red edges. By definition of N_{red}, the clique induced by T has either a $(p-1)$ red clique (combining with x to form a p-clique) or a q blue clique. In either case the Ramsey number is achieved. Symmetric discussion for M (blue neighbors). $\square$

12.2 General Ramsey Theorem

Theorem 0.52 (10.2.5 Ramsey's Theorem)**.** For $k, r, p_1, \dots, p_k \in \mathcal{N}$, the Ramsey number $R(p_1, \dots, p_k; r)$ exists.

Proof. Induction on r. For fixed r, use induction on p_i.

Base case ($r = 1$): Coloring of $\binom{S}{1}$, which is just vertex coloring. By Pigeonhole Principle, we get $N = p_1 + \cdots + p_k - k + 1$.

For $r > 1$: (i) If some $p_i < r$, let $|T| = p_i < r$, then $\binom{T}{r}$ is empty, thus no r-sets of T.

Use T for the set is always i-homogeneous.

$R(p_1, \dots, p_k; r) = \min(p_1, \dots, p_k)$ if $\min p_i < r$. (You only need to color the vertices of T then the requirement is satisfied.)

(ii) We prove the situation for $k = 2$, other k situations follow. (For $k = 2$, use intuition from 2-edge coloring). Write $R(p, q)$ for $R(p, q; r)$ (Red and Blue). Let $p' = R(p - 1, q; r)$, $q' = R(p, q - 1; r)$. Let $N = 1 + R(p', q'; r - 1)$.

Remark 0.34 (Intuition)**.** p' is large enough s.t. inside any set of size p' with r-subset coloring, either exists a red-homo $(p - 1)$-set or a blue-homo q-set. $R(p', q'; r - 1)$ is threshold for $(r - 1)$-subset coloring, guarantees either red-homo p'-set or blue-homo q'-set. Add 1 for x.

Let S s.t. $|S| = N$. Pick $x \in S$, $S' = S \setminus \{x\}$. $|S'| = R(p', q'; r - 1)$.

Suppose f is a 2-coloring of $\binom{S}{r}$. Need to show there exists a red-homo p-set or a blue-homo q-set.

Based on f, define f' s.t. $f'(A) = f(A \cup \{x\})$ for any $(r - 1)$-subset $A \subseteq S'$.

Since $|S'| = R(p', q'; r - 1)$, apply Induction Hypothesis guarantees either a red-homo p'-set or a blue-homo q'-set under f'.

WLOG Assume red quota is met.

Recall definition of f': every r-set that contains x and whose other $r - 1$ elements lie in T is red in the original coloring f.

Return to f on $\binom{T}{r}$. Since $|T| = p' = R(p - 1, q; r)$, apply inner induction hypothesis on $p + q$ (p_i), we conclude R exists, which means either a red-homo $(p - 1)$-set under f or a blue-homo q-set.

If blue set exists we are done since it's a subset of S and it's homogeneous.

If red set exists, claim the set together with x is the expected p red-homo set.

Indeed, call this $(p - 1)$-set P. Take any r-subset B of $P \cup \{x\}$, we need to show B is red-homo under f.

- If $x \notin B$, $B \subseteq P$, P is red-homogeneous under f and we are done.
- If $x \in B$, $B = A \cup \{x\}$ for some A $(r - 1)$-subset. But we choose T s.t. all $(r - 1)$-subsets are red-homogeneous under f'. And by definition of f', $f'(A) = \text{red} \iff f(A \cup \{x\}) = \text{red}$. Which means $f(B) = \text{red}$, and we are done.

Therefore, r-subset B of $P \cup \{x\}$ is always red, hence $P \cup \{x\}$ is red-homo of size p. □

12.3 Applications of Ramsey

Theorem 0.53 (10.2.6 Erdos-Szekeres)**.** For $m \in \mathcal{N}$, there is at least integer $N(m)$ s.t. every set of $N(m)$ points in the plane contains an m-subset forming a convex m-gon.

Proof. **Fact 1:** Among any 5 points, 4 form a convex quadrilateral.

Fact 2: If every 4-subset of m points is a convex quadrilateral, then all m points are in convex position.

Then, set $N = R(m, 5; 4)$. (This is Ramsey number for 4-uniform hypergraph with 2 colors).

Find either an m-subset whose 4-sets are all red, or 5-subset whose 4-sets are all blue.

Color the N points' 4-subsets:

(i) Red if they form convex quadrilateral.

(ii) Blue otherwise.

Then among any 5-points, there exists some red 4-subset by Fact 1, so there's no blue-homogeneous 5-set.

Thus the red-homo condition must be satisfied, i.e., every 4 points in an m-subset M forms a convex quadrilateral.

But then by Fact 2, the whole m-set M is in convex position, which means it forms a convex m-gon.

Therefore $N(m)$ exists, and even more $N(m) \leq R(m, 5; 4)$. □

13 Schur's Theorem

Theorem 0.54 (10.3.1 Schur's Theorem)**.** Given $k > 0$, there is an integer s_k s.t. every k-coloring of $\{1, \dots, s_k\}$ yields monochromatic (not necessarily distinct) x, y, z solving $x + y = z$.

Proof. Let $r_k = R_k(3; 2)$ (the least N s.t. every k-coloring of the edges of the complete graph K_N contains a monochromatic triangle).

Show $s_k \leq r_k$ by showing every k-coloring f of $[r_k - 1]$ has a monochromatic solution to $x + y = z$.

Define a k-coloring f' of $E(K_{r_k})$. $f'(i, j) = f(|i - j|)$.

By definition of r_k, f' gives a monochromatic triangle with some vertices a, b, c.

WLOG $a < b < c$.

Let $x = b - a, y = c - b, z = c - a$.

Then $f'(ab) = f'(bc) = f'(ac) \implies f(b - a) = f(c - b) = f(c - a)$.

But $x + y = c - a = z$, (x, y, z) is a monochromatic solution to $x + y = z$.

Thus $r_k - 1$ is a valid option for s_k, i.e., $s_k \leq r_k - 1$. And it exists by existence of Ramsey number. □