From Domino Puzzles to Math and Computer Science

Youyou Wu

Department of Mathematics
University of Illinois, Urbana-Champaign

Illinois CS Sail lecture

UIUC

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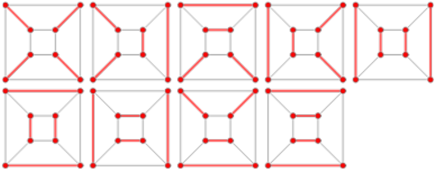

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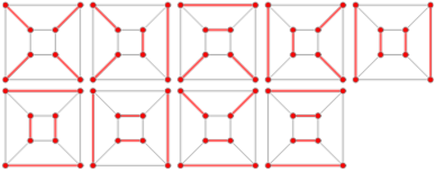

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- I do research in graph theory. I also know a decent amout of academic resources around campus, so feel free to reach out! An important part of college life is also learning to network with people.

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Background & Motivation

What is a tiling problem? (well actually it's hard to define...)

Definition (informal!)

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- Historically, people have been studying domino tiling problem by counting number of perfect matchings in m*n grid graphs. Methods is extemely rich and complicated.
- However, using the language of tiling, most of the results can in fact be obtained with elementary proofs, and is the main focus for us today.

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Proof.

proof by picture (We love pictures!) □

Taped Bracelet

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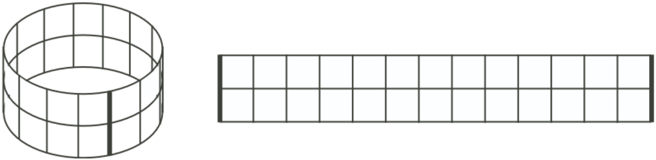

Figure 2: A 2×13 taped bracelet grid along with a more convenient representation thereof.

Taped Bracelet

Theorem

*Let B_n be the number of ways to tile a 2*n taped bracelet grid with 2*1 tiles. Then*

$$B_n = R_n + R_{n-2} + 2((n-1) \ mod\ 2) = B_{n-1} + B_{n-2} - 2(n \ mod\ 2).$$

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Figure 3: A 2 × 4 taped bracelet grid, partially tiled and with half of the tape line covered with a horizontal tile.

Figure 4: The eight possible neighborhoods of the tape on a tiled 2 × n taped bracelet (or Möbius) grid.

Taped Cylinder

What if we tape the top and bottom?

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Definition

A **taped cylinder** is obtained by bending a 2*n grid so that the top and the bottom meet.

Taped Cylinder

Theorem

*Let C_n be the number of ways to tile a 2*n taped cylinder grid with 2*1 tiles. Then*

$C_n = 2C_{n-1} + C_{n-2}$. *Notice that this is the* **Pell sequence.**

Proof.

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Figure 5: The two ways to place a vertical tile on a taped cylinder grid.

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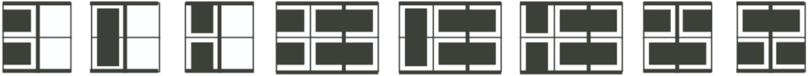

Figure 6: The eight possible neighborhoods of the tape on a tiled $2 \times n$ taped torus (or Klein) grid.

Taped Mobius Band

Ready for some challenge?

Taped Mobius Band

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Definition

A **Taped Mobius Band** is obtained by first bending a 2*n grid into a taped bracelet, but then give one side a **half-twist** before securing both ends together with a tape.

Taped Mobius Band

Theorem

*Let M_n be the number of ways to tile a 2*n taped Mobius grid with 2*1 tiles. Then*

$M_n = R_n + R_{n-2} + 2(n \ mod \ 2) = M_{n-1} + M_{n-2} - 2((n-1) \ mod \ 2).$

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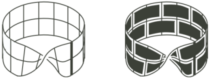

Figure 7: A 2×13 taped Möbius grid, untiled (left) and tiled with an odd-only tiling (right).

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Definition (Mutilated chessboard problem)

Suppose a standard 8×8 chessboard has two diagonally opposite corners removed, leaving 62 squares. Is it possible to place 31 dominoes of size 2×1 so as to cover all of these squares? [Wikipedia]

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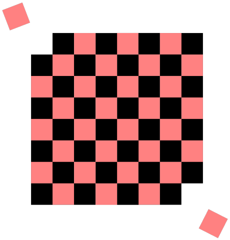

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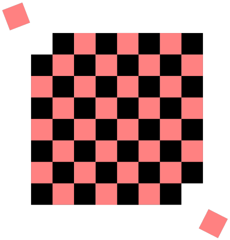

Try to come up with the answer yourself!

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- After removing the two red blocks, we have 30 red blocks and 32 black blocks remaining.
- But this means after placing 30 dominos, we will use up 30 red blocks and 30 black blocks, leaving 2 black blocks untiled; but since dominos can only occupy opposite colors, we can not tile the left two blocks.

□

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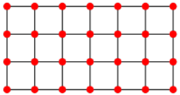

Figure: Grid Graph

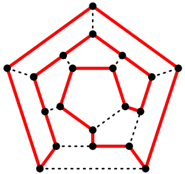

Figure: Hamiltonian Cycle

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It turns out, the proof depends on 3 sufficient conditions:

- One side of the rectangle has to be even.
- When deleting the blocks of opposite colors, their numbers have to be equal, and their order has to be **alternating** along the hamiltonian cycle!

Questions?